A MULTIVARIATE STATISTICAL MODEL FOR ESTIMATING THE NUMBERS, SIZES, AND INTERFLUVIAL DISTANCES OF STREAMS WITHIN A RELATIVELY HOMOGENEOUS DRAINAGE AREA

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THESIS

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ABSTRACT

A model has been developed for estimating the number, sizes, and interfluvial distances of streams to be crossed in traversing a naturally occurring drainage basin. A relationship among four network parameters has been found which, upon quantification, provide sufficient information to completely structure a mature drainage system. The four parameters are: stream length ratio, bifurcation ratio, basin shape parameter, and coefficient of drainage density. The model is amenable to computer solution. Procedures are described for parameter quantification using common topographic maps.

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I. SUMMARY

In order to determine requirements for military stream crossing equipment and materiel, it is necessary to determine the number and sizes of streams (or gaps) that have to be crossed. The task of actually counting and measuring streams throughout large areas of the world either by reconnaissance, from topographic maps, or by other means such as aerial photographs, is fraught with difficulties of time, money, and access. Consequently, a model is needed that will circumvent these difficulties and provide reliable data on stream size and frequency in a manner sufficiently timely to facilitate tactical and/or strategic military planning.

This thesis presents such a model, based upon previous theoretical and empirical work in the field of quantitative geomorphology. A number of relationships are developed that were not heretofore available. The model combines two methods of stream ordering to describe a drainage basin in terms of its component streams and basins. The model relies on quantification of four parameters to estimate the frequency and sizes of streams within a naturally occurring drainage area. The four parameters are: stream ratio (α), bifurcation ratio (β), basin shape parameter (h), and coefficient of drainage density (k).

Reliability of the model is materially enhanced by quantifying all four parameters within the geographical

area of intended model use. However, the model reliability is little reduced if h and k are assumed to have equilibrium values and only α and β are quantified within the area of intended model use. Consequently, employment of the model would be greatly facilitated if topographic maps of the appropriate areas were overprinted with the applicable α and β values.

The model does not attempt to predict the hydrologic characteristics of streams such as depth, velocity, width, etc., for the different stream orders. Such a model is a desirable adjunct to the model presented by this paper and offers fertile ground for further research. A model of this general nature that could be used was formulated by Mayer [Ref. 32], however, further study needs to be done regarding the essential features of bank conditions and the effects of the works of man such as bridges, lock and dam restrictions, etc. It is believed, nonetheless, that the model presented by this paper is an essential step in obviating detailed ground, map, and aerial reconnaissance for large-scale planning and capabilities studies.

II. INTRODUCTION

With the possible exception of the enemy, perhaps the greatest single obstacle to cross-country movement of a combat force has traditionally been the naturally occurring stream. In fact, the location and obstacle characteristics of streams have on more than one occasion practically dictated the nature of the tactical plan. But, with the state of modern stream crossing technology it is more instructive to think of such obstacles in terms of the relative degree to which they impede the speed of movement of the combat force. In this context the problem assumes more general characteristics and has been properly labelled the "gap crossing problem."

A gap is defined as any cavity in the earth's surface caused by flowing water. Thus, the study of gaps is necessarily the study of streams. Streams are of many shapes and forms but may be generally classed as ephemeral, intermittent, or perennial. An ephemeral stream flows only during or immediately following periods of precipitation and handles only surface runoff. An intermittent stream flows only during the wet season and differs little from ephemeral streams. A perennial stream, on the other hand, flows constantly. The perennial stream is generally considered to have the greatest obstacle value. However, with modern stream crossing means, other aspects of the streams are often as important as the presence or absence of water.

In any event, streams (or gaps) have sufficient obstacle value that no tactical plan involving the cross-country movement of troops is complete without a detailed consideration of the streams within the zone of movement. This has traditionally been accomplished by examining the area to be traversed by personal reconnaissance, examining topographic maps, or, more recently, aerial photographs or electronic means. However, it is not difficult to visualize circumstances under which neither method will produce satisfactory results. For example, personal reconnaissance in prospective enemy territory during peacetime is quite difficult in addition to being quite limited in scope. Further, topographic maps of the tactical variety seldom, if ever, provide sufficient detail for the smaller gaps. Aerial photographs and the other, electronic, means suffer from these shortcomings plus being quite expensive and time-consuming to obtain and interpret. In short, a better method is needed for determining (or estimating) the number and sizes of gaps within a given area, a method that is inexpensive, timely, and widely applicable. This model, then, seeks to provide that method by describing the behavior of naturally occurring stream networks in mathematical terms and using the theory of the most probable state to secure useful predictive results.

The study of the formation of drainage patterns from a stochastic viewpoint had an early beginning. In 1802, John Playfair, Professor of Natural Philosophy, University of

Edinburgh, wrote Illustrations of the Huttonian Theory of the Earth. The following passage has been called Playfair's Law:

> Every river appears to consist of a main trunk, fed from a variety of branches, each running in a valley proportioned to its size, and all of them together forming a system of valleys, communicating with one another, and having such a nice adjustment of their declivities that none of them join the principal valley either on too high or too low a level; a circumstance which would be infinitely improbable if each of these valleys were not the work of the stream which flows in it.

Although Playfair obviously recognized the systematic aspects of a natural drainage system, little was done to quantify these aspects prior to the work of R. E. Horton in 1945 [Ref. 11]. Horton's bifurcation and stream length ratios are particularly important to this model and to most of the work that has been accomplished in the field since Horton's publication. In recent years probability theory has been used in an attempt to explain and quantify various aspects of naturally occurring drainage networks. Prominent among the works in this area have been those by Hack [Ref. 27], Leopold and Langbein [Ref. 28], and Schenck [Ref. 18]. The general theory basic to these works is that the formation of drainage networks can be explained by the laws of probability and that, free of constraints, streams occur in a random fashion. Leopold and Langbein viewed the laws of Horton as representing the most probable state in a stochastically formed network, and used a random walk

model to construct a network representing the most probable state to show that the laws of Horton did, in fact, hold. Schenck used a similar model to substantiate the empirical work of Gray [Ref. 9]. The first attempt at constructing a predictive model was made by Hugo E. Mayer [Ref. 32]. The Mayer model was an attempt to use the work of the several previous authors in the field to construct drainage basin relationships that, with quantification of a few parameters, could be used to predict the numbers, orders, and hydrologic characteristics of streams within a naturally occurring drainage network. The model presented by this thesis has some of the characteristics of the Mayer model, but is designed to be a more sophisticated and a more exact presentation.

This thesis will seek to build upon accepted laws and, using these laws and other considerations, to establish a number of relationships not heretofore available. The laws of Horton [Ref. 11] and Schumm [Ref. 34] will be used essentially as presented by these authors with an additional law to be deduced from the empirical work of Hack [Ref. 27]. The first effort toward model construction will be to define an admissable region for the key model parameters. The polynomials describing the upper and lower bounds for the region were defined by Mayer [Ref. 32]. The polynomial describing equilibrium behavior within the region will be developed by making certain assumptions concerning the

"jumping" process and then describing in mathematical terms the proportion of streams of a given order that jump and, of those that jump, the proportion that jumps to a specific order. The product of the latter two proportions produces the proportion of streams of order i that jump to order j for all i = 1, 2, ---, N, and all j > i. In particular, it will facilitate describing the proportion of streams within a drainage area of order i that jumps to order N, the order of the parent stream. This relationship employed in conjunction with an equation by Hack [Ref. 27] will be used to write the polynomials describing the admissable region for the model parameters. This admissable region will be used to estimate parameter values within drainage areas for which specific parameter quantification is not practical and to estimate the error to be expected from such estimates. In defining the admissible region a number of assumptions will be necessary. The thesis will attempt to evaluate the impact of these assumptions on ultimate model output and to establish relatively simple relationships that demonstrate the logic underlying the assumptions. Finally, the model will be given a statistical structure by assuming distribution functions for the relevant variables. A uniform stream distribution within the drainage basin will be used to produce a sample solution, however, with relatively minor modifications the model will accommodate other distribution functions. Mean conditions will be used to write equations describing the various aspects of importance

to this model and the confidence to be associated with each estimate. A five-percent perturbation will be used to demonstrate the sensitivity of the model to its various parameters and to establish those parameters for which specific area quantification is necessary for a high measure of model reliability. The latter will facilitate recommendations concerning exogenous preparations required for effective model employment.

III. BASIC RELATIONSHIPS

A. NOTATION

α = stream length ratio

β = bifurcation ratio

ρ = overland flow ratio

λ = basin area ratio

h = basin shape parameter

D = drainage density

k = coefficient of drainage density

N = order of the main stream of the parent basin

i = an index denoting stream order ($i = 1,2,---,N$)

j = an index where $j = N - i$ ($j = 0,1,---,N-1$)

P_i = proportion of ith - order streams that jump

P_{ir} = proportion of ith - order streams that jump to streams of order r

$\bar{a}_i$ = mean area of ith - order basins, square miles

a_{ki} = area of the kth basin of ith order, square miles

A_i = total area of all ith - order basins, square miles

n_i = number of ith - order streams

ℓ_{ik} = length of the kth stream of ith order, miles

$\bar{\ell}_i$ = mean length of all ith - order streams, miles

L_N = length of the parent basin, miles

d_i = mean interfluvial distance, i.e., distance between streams of ith order, miles

W_N = width of the parent basin, miles

f_i = mean area drained by overland flow occurring directly into an ith - order stream, square miles

F_i = total area drained by overland flow occurring directly into an ith - order stream, square miles

η_i^* = the number of ith - order streams encountered by a force in one traverse of a mean basin along the minor axis

η_i^{**} = the number of ith - order streams encountered by a force in one traverse of a mean basin along the major axis

Φ_i^* = expected total length of ith - order streams encountered by a force in one traverse of a mean basin along the minor axis, miles

Φ_i^{**} = expected total length of ith - order streams encountered by a force in one traverse of a mean basin along the major axis, miles

B. DEFINITIONS

1. Drainage Basin. The area of slopes which lead into a single stream system

2. Drainage Divide. The line of separation that divides the precipitation that falls on two adjoining basins and directs the ensuing runoff into one or the other river system. The definition of drainage divide used in this thesis will refer to the topographic or surface divide as distinguished from the phreatic or underground divide

3. Stream Order. The stream order is a measure of the stream's rank in the hierarchy of streams that constitute the drainage network

4. Basin Order. The hierarchal order of a basin is the same order as the highest order stream in the basin

5. Parent Basin. The parent basin is the drainage basin containing the highest order stream, and all lower

order tributaries with their associated basins, of the drainage system being studied

C. STREAM ORDERING METHODS

In developing this model two methods of stream ordering will be used. The one presented by Strahler [Ref. 23] will be used to develop the bifurcation ratio and the one presented by Horton [Ref. 11] will be used to develop the stream length ratio. Both methods are illustrated in Figure 1-1.

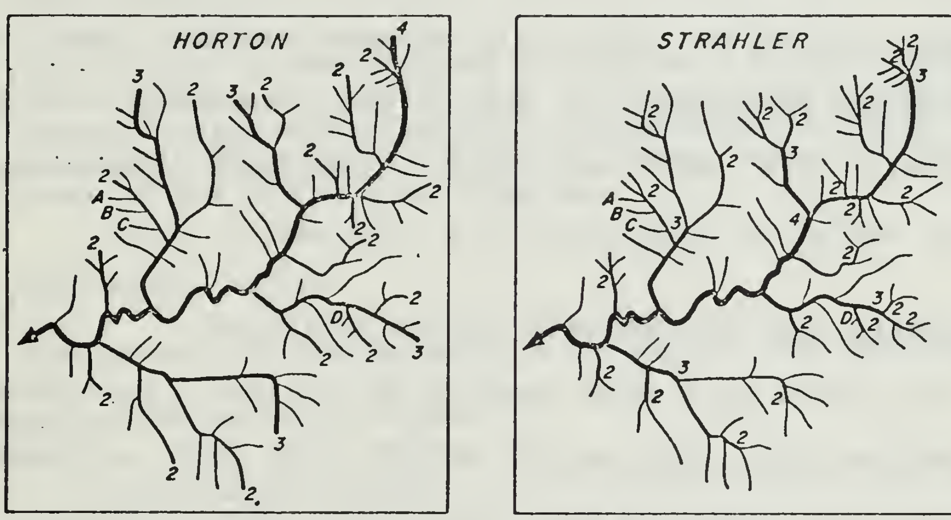


Figure 1-1. Horton and Strahler definitions of stream order applied to channel network of Hightower Creek, upper Hiwassee River, Towns County, northern Georgia (extracted from Shreve [Ref. 19]). Order indicated by number near upstream end of respective streams. Unnumbered streams are first order. If streams A, B, C, or D were actually second order, rather than first, then network would be fifth order, rather than fourth.

Comparison of the two methods as illustrated facilitates a rapid differentiation of the results provided by the different stream ordering methods. The Horton method traces each stream to its drainage divide and considers that each stream extends from its mouth to primary headwater. Thus, it is intuitively appealing to associate the stream length ratio with this method. The Strahler method, on the other hand, defines a first order stream as one having no tributaries and provides that when two first order streams intersect, a second order stream is formed, that the intersection of two second order streams forms a third order stream, and so on. Thus, the streams are not traced back to primary headwater but only to the intersection of two streams of the next lower order. The latter system proves quite useful in developing the bifurcation ratio.

D. BASIC LAWS OF DRAINAGE SYSTEM EVOLUTION

The model will draw heavily on two basic laws presented by Horton [Ref. 11], one by Schumm [Ref. 34], and another derived from works by Hack [Ref. 27]. The Horton laws are:

> (1) Law of Stream Numbers: The numbers of streams of different orders in a given drainage basin tend closely to approximate an inverse geometric series in which the first term is unity and the ratio is the bifurcation ratio.
>
> (2) Law of Stream Lengths: The average length of streams of each of the different orders in a drainage basin tend closely to approximate a direct geometric series in which the first term is the average length of streams of the first order [Ref. 11:291].

The Schumm law of stream areas states:

> the mean drainage basin areas of streams of each order tend closely to approximate a direct geometric series in which the first term is the mean area of the first order basins [Ref. 34:14].

Hack observed that overland flow contributes some runoff directly into streams of all orders. The length of runoff directly into streams of order greater than one is, of course, insufficient to sustain a first order stream. Otherwise, a first order stream would be formed. However, Hack observed that the area drained in this manner was proportional to the stream length ratio. Hack's observation implies our final law of drainage evolution which can be expressed as:

> the average areas drained by overland flow directly into streams of each of the different orders within a parent basin tend closely to approximate a direct geometric sequence in which the first term is the mean area of the first order basins.

E. THE MODEL RATIOS

The foregoing laws enable us to define the following dimensionless relationships.

1. Stream Length Ratio. The stream length ratio is the ratio of the average length of streams of a given order to the average length of streams of the next lower order:

$$\alpha = \frac{\bar{\ell}_i}{\bar{\ell}_{i-1}}, \; i > 1. \tag{1.1}$$

2. Bifurcation Ratio. The bifurcation ratio is the ratio of the number of streams of a given order to the number of streams of the next higher order:

$$\beta = \frac{n_i}{n_{i+1}}\ , \ i < N\ . \tag{1.2}$$

3. Overland Flow Ratio. The overland flow ratio is the ratio of total area drained by overland flow directly into the streams of a given order to the total area drained by overland flow directly into the streams of the next lower order:

$$\rho = \frac{F_i}{F_{i-1}}\ , \ i > 1\ . \tag{1.3}$$

4. Basin Area Ratio. The basin area ratio is the ratio of the total area of basins of a given order to the total area of the basins of the next higher order:

$$\lambda = \frac{A_i}{A_{i+1}}\ , \ i < N\ . \tag{1.4}$$

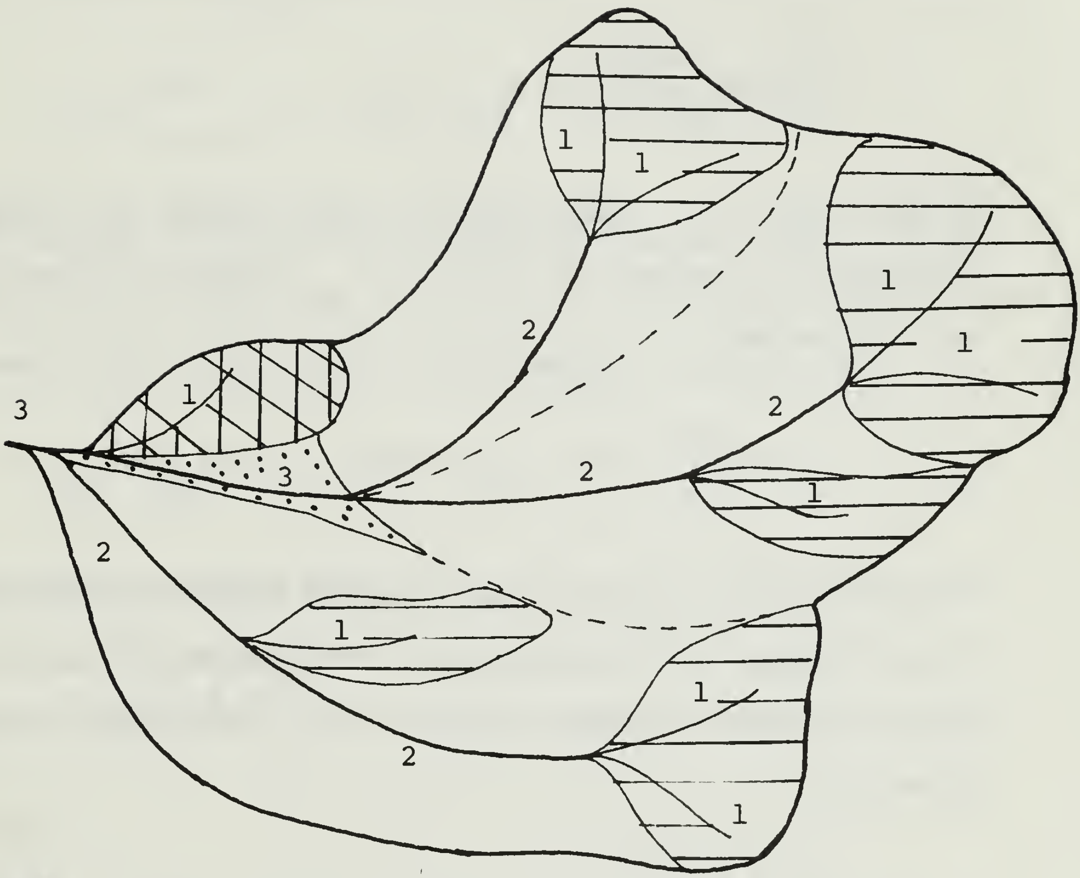
A hypothetical third order basin using the Strahler method of stream ordering is illustrated in Figure 1-2.

F. A DESCRIPTION OF THE BASIC LAWS

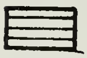
1. Horton's Laws. Horton's first law can be written as

$$\{\beta^i\}_{i=0}^{N-1} = 1,\ \beta,\ \beta^2,\ \ldots,\ \beta^{N-1}\ ,$$

where the first term is the number of streams of order N and the last term is the number of streams of order 1. Thus, the number of streams of order i can be written as




First order basin not included in a parent second order basin


First order basin included in a parent second order basin


Second order basin portion not included in a first order basin (overland flow directly into second order stream)

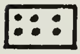

Third order basin portion not included in a first or second order basin (overland flow directly into third order stream)

Figure 1-2. Basins of Hypothetical Third Order Stream (Redrawn from Mayer [Ref. 32])

$$n_i = \beta^{N-i} \quad . \qquad (1.5)$$

If we form the bifurcation ratio

$$\frac{n_i}{n_{i+1}} = \frac{\beta^{N-i}}{\beta^{N-i-1}} = \beta \quad ,$$

we see that β is independent of basin order and thus can be considered constant for all $i < N$. Horton's second law can be written as

$$\{\bar{\ell}_1 \alpha^i\}_{i=0}^{N-1} = \bar{\ell}_1, \ \bar{\ell}_1\alpha, \ \bar{\ell}_1\alpha^2, \ \ldots, \ \bar{\ell}_1\alpha^{N-1},$$

where the first term is the average length of all first order streams and the last term the length of the Nth - order stream. Thus, the mean length of an ith - order stream is given by

$$\bar{\ell}_i = \bar{\ell}_1\alpha^{i-1} \qquad (1.6)$$

where

$$\bar{\ell}_i = \frac{1}{n_i}\sum_{k=1}^{n_i} \ell_{ik} \quad .$$

If we form the stream length ratio

$$\frac{\bar{\ell}_i}{\bar{\ell}_{i-1}} = \frac{\bar{\ell}_1\alpha^{i-1}}{\bar{\ell}_1\alpha^{i-2}} = \alpha \quad ,$$

we see that α is also independent of basin order and can be considered constant for all $i > 1$.

2. Schumm's Law. Schumm's law of stream areas can be written symbolically as

$$\{\bar{a}_1 \theta^i\}_{i=0}^{N-1} = \bar{a}_1, \bar{a}_1\theta, \bar{a}_1\theta^2, \ldots, \bar{a}_1\theta^{N-1} ,$$

where the first term is the mean area of all first order basins and the last term the area of the parent basin. Thus, the mean area of the ith - order basin is given by

$$\bar{a}_i = \bar{a}_1 \theta^{i-1} ,$$

which can be written as

$$A_i = \frac{n_i}{n_1} A_1 \theta^{i-1} .$$

Note that

$$A_i = \sum_{k=1}^{n_i} a_{ki} = n_i \bar{a}_i .$$

If we form the basin area ratio

$$\frac{A_i}{A_{i+1}} = \frac{n_i A_1 \theta^{i-1} n_1}{n_{i+1} A_1 \theta^i n_1} = \beta\theta^{-1} = \lambda ,$$

we see that λ is independent of basin order and thus constant for all $i < N$. By solving recursively we obtain the more useful relationship

$$A_{N-j} = \lambda^j A_N \; . \tag{1.7}$$

3. <u>The Law of Overland Flow Areas</u>. The law of overland flow areas can be written symbolically as

$$\{\bar{a}_1\alpha^i\}_{i=0}^{N-1} = \bar{a}_1,\ \bar{a}_1\alpha,\ \bar{a}_1\alpha^2,\ \ldots,\ \bar{a}_1\alpha^{N-1} \; ,$$

where the first term is the mean area of all first order streams and the last term the area drained by overland flow occurring directly into the Nth - order stream. Thus, the area drained by overland flow directly into an ith - order stream can be written as

$$f_i = \alpha^{i-1}\bar{a}_1 \; .$$

If we further define F_i as the total area drained by overland flow directly into an ith - order stream, we can write

$$F_i = n_i\bar{a}_1\alpha^{i-1} \; .$$

Forming the overland flow ratio we obtain

$$\frac{F_i}{F_{i-1}} = \frac{n_i\bar{a}_1\alpha^{i-1}}{n_{i-1}\bar{a}_1\alpha^{i-2}} = \alpha\beta^{-1} = \rho,$$

which is constant within a relatively homogeneous drainage basin since by Horton's first and second laws, α and β are constants. Solving recursively we obtain

$$F_i = F_1\rho^{i-1} \; ,$$

and since $F_1 = A_1$, we further obtain

$$F_i = A_1 \rho^{i-1} .$$

Finally, using equation 1.7 with $i = N-j$, we obtain the useful relationship

$$F_i = (\rho\lambda)^{i-1} A_i \quad . \tag{1.8}$$

G. CONVERSION BETWEEN STREAM ORDERING METHODS

Since the Horton method of stream ordering extends the streams back to primary headwater, we must convert the stream lengths to the Strahler method in order to make them compatible with the bifurcation ratio. An examination of Figure 1-1 reveals that to accomplish this conversion requires only that we subtract from each Horton mean stream length of a given order a Horton mean stream length of the next lower order. Symbolically, if we define $\overline{\ell}_i = \overline{\ell}_1 \alpha^{i-1}$ using the Horton method, then the corresponding mean stream length using the Strahler method can be written as

$$\overline{\ell}_i = \overline{\ell}_1 (\alpha^{i-1} - \alpha^{i-2}),\ i > 1. \tag{1.9}$$

Forming the stream length ratio we obtain

$$\frac{\overline{\ell}_i}{\overline{\ell}_{i-1}} = \frac{\overline{\ell}_1 (\alpha^{i-1} - \alpha^{i-2})}{\overline{\ell}_1 (\alpha^{i-2} - \alpha^{i-3})} = \alpha,$$

which reveals that the stream length ratio is unchanged by the conversion and, thus, the two stream ordering methods can be used interchangeably as far as the stream length ratio is concerned. The numbers of streams of various orders can also be quickly converted from one method to the other. If we denote by n_{jH} the number of streams of order j using the Horton method and by n_{jS} the number of streams of order j using the Strahler system, then

$$n_{jS} = \sum_{i=j}^{N} n_{iH} , \quad j = 1, 2, \ldots, N,$$

and, conversely,

$$n_{jH} = n_{jS} - n_{j+1,S} , \quad j = 1, 2, \ldots, N-1 .$$

Thus, as indicated by Shreve [Ref. 19], the number of first order Strahler streams is equal to the total number of Horton streams within the parent basin. However, the bifurcation ratio is unchanged and, thus, the two methods are again interchangeable with respect to the corresponding dimensionless ratio.

IV. FUNDAMENTAL PROCESSES

A. JUMPING

We hypothesize that within an Nth - order parent basin streams of order i bifurcate with streams of order i+1, i+2, ---, N, in direct proportion to the relative total stream length of the recipient streams. Note that a stream of order i cannot bifurcate with any order less than i+1; therefore, relative stream length refers to streams of orders i+1 and greater. This hypothesis is supported by considerations of probability in a relatively homogeneous drainage basin. As indicated by Leopold and Langbein [Ref. 28] in their concept of entropy in landscape evolution, rainfall occurring on a uniformly sloping surface is as likely to produce a stream of a given order on one part of the surface as on another and the orders of bifurcating streams are largely determined by chance. Thus, it follows that first order streams, for example, will intersect third order streams in direct proportion to the relative total length of all third order streams within the drainage basin, where relative stream length refers only to those streams of order greater than one. Under the Strahler system, at least two ith - order streams are necessary to form each (i+1)st - order stream. Therefore, we can express the number of ith - order streams that bifurcate with (i+1)st - order streams as

$$n_{i,i+1} = 2n_{i+1} + (n_i - 2n_{i+1}) \frac{n_{i+1}\overline{\ell}_{i+1}}{\sum_{j=i+1}^{N} n_j \overline{\ell}_j} .$$

Dividing this by n_i we can write the proportion of ith-order streams that bifurcate with (i+1)st-order streams as

$$\frac{n_{i,i+1}}{n_i} = \frac{2n_{i+1}}{n_i} + \frac{(n_i - 2n_{i+1})}{n_i} \frac{n_{i+1}\overline{\ell}_{i+1}}{\sum_{j=i+1}^{N} n_j \overline{\ell}_j} .$$

Since the proportion of ith-order streams that jump is the complement of the proportion that bifurcates with streams of the next higher order, we can write the proportion that jumps as

$$P_i = 1 - \frac{2n_{i+1}}{n_i} - \frac{(n_i - 2n_{i+1})}{n_i} \frac{n_{i+1}\overline{\ell}_{i+1}}{\sum_{j=i+1}^{N} n_j \overline{\ell}_j} ,$$

which, using equations 1.5 and 1.6, reduces to

$$P_i = \frac{(\rho - \rho^{N-i})(1 - 2\beta^{-1})}{1 - \rho^{N-i}} , \quad i < N . \qquad (2.1)$$

Equation 2.1 describes the behavior of P_i within certain bounds. The lower bound is trivial and the upper bound is determined from the fact that at least two ith-order streams are required to constitute each (i+1)st-order stream. The bounds on P_i can be written as

$$0 \leq P_i \leq \frac{n_i - 2n_{i+1}}{n_i} ,$$

which can be further expressed as

$$0 \leq P_i \leq (1 - 2\beta^{-1}) .$$

The behavior of P_i in a tenth-order basin wherein α and β are arbitrarily chosen to be 2.68 and 4, respectively, is illustrated in Table 2-1. The table also gives the proportion of streams that don't jump as q_i. In evaluating the table contents note the important characteristic that the only stream permitted to exit a parent basin is the Nth-order stream. All other streams lie entirely within the Nth-order basin.

TABLE 2-1

BIFURCATION BEHAVIOR OF INCLUDED STREAMS WITHIN A HYPOTHETICAL TENTH-ORDER DRAINAGE BASIN

ORDER (i)	PROPORTION THAT JUMP (P_i)	PROPORTION THAT DON'T JUMP (q_i)
1	.336	.664
2	.333	.667
3	.329	.671
4	.323	.677
5	.314	.686
6	.297	.703
7	.267	.733
8	.203	.797
9	0	1.000

Now, of the ith-order streams that jump, we define P_{ir} as the proportion that jumps specifically to order r. The process of jumping within a parent basin is then as illustrated in Figure 2-1. Further, we can write, symbolically, the proportions that jump to order N as

$$P_{N-2,N} = \frac{\bar{\ell}_N n_N}{\bar{\ell}_N n_N} = 1 ,$$

$$P_{N-3,N} = \frac{\bar{\ell}_N n_N}{\bar{\ell}_N n_N + \bar{\ell}_{N-1} n_{N-1}} = \frac{1}{1 + \rho^{-1}} = \frac{1 - \rho^{-1}}{1 - \rho^{-2}} ,$$

$$\vdots$$

and $$P_{N-j,N} = \frac{\bar{\ell}_N n_N}{\bar{\ell}_N n_N + \ldots + \bar{\ell}_{n-j+2} n_{n-j+2}} = \frac{1}{1 + \rho^{-1} + \ldots + \rho^{2-j}}$$

$$= \frac{1 - \rho^{-1}}{1 - \rho^{1-j}} .$$

The proportion of all streams of order i that jump to order N can now be written as

$$P_i P_{iN} = \frac{(\rho - \rho^{N-i})(1 - 2\beta^{-1})}{1 - \rho^{N-i}} \cdot \frac{1 - \rho^{-1}}{1 - \rho^{1-N+i}}$$

$$= \frac{(\rho^{N-i-1} - \rho^{N-i})(1 - 2\beta^{-1})}{1 - \rho^{N-i}} . \qquad (2.2)$$

B. THE BASIC PARAMETRIC RELATIONSHIP

Schumm [Ref. 34] added to his law of stream areas "It could be assumed that such a relationship would exist if

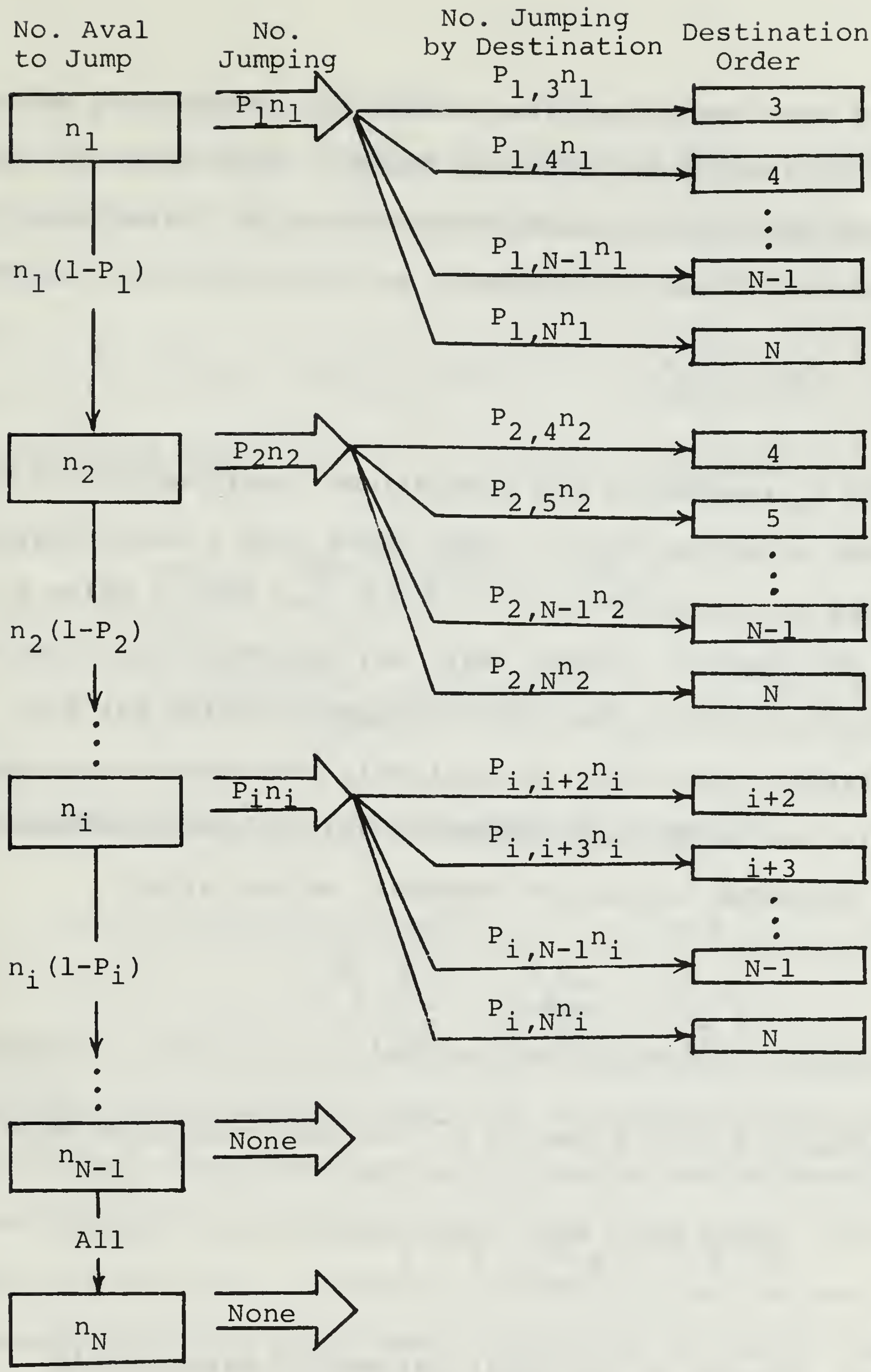


Figure 2-1. The Jumping Process

there were any connection between the length of a stream and the size of its drainage basin." Hack [Ref. 27] later showed that such a relationship did exist. The Hack relationship can be expressed as

$$\bar{\ell}_i = k\bar{a}_i^{\,h} \ , \tag{2.3}$$

where $\bar{\ell}_i$ represents the mean stream length using the Horton stream ordering method. Hack found that k could assume values in the range $1.0 \leq k \leq 2.5$, but that a value of 1.4 was typical. Mayer [Ref. 32] observed that h was bounded by $0.5 \leq h \leq 1.0$, but that the typical value was 0.6. If we assume that within a relatively homogeneous drainage basin the basin shape parameter (h) and the coefficient (k) for included basins are constant, we can write

$$\alpha = \frac{\bar{\ell}_i}{\bar{\ell}_{i-1}} = \frac{k\bar{a}_i^{\,h}}{k\bar{a}_{i-1}^{\,h}} = \left[\frac{\bar{a}_i}{\bar{a}_{i-1}}\right]^h \ .$$

Multiplying both sides by β^{-1} and rearranging we obtain

$$\beta^{-1}\alpha^{1/h} = \frac{A_i}{A_{i-1}} = \lambda^{-1} \ ,$$

which reduces to the basic parametric relationship

$$\beta = \lambda\alpha^{1/h} \ . \tag{2.4}$$

The constancy of h and k are discussed more fully in Chapter III.

C. THE FUNDAMENTAL EQUATION

Considering both jumping and overland flow, we can now account for all of the area composing a parent basin. In symbolic form the area is

$$A_N = A_{N-1} + P_{N-2}P_{N-2,N}A_{N-2} + \cdots + P_1 P_{1,N} A_1 + F_N ,$$

which, using equations 1.7, 1.8, and 2.2, becomes

$$A_N = \lambda A_N + \left[\frac{(\rho-\rho^2)(1-2\beta^{-1})}{1-\rho^2}\right] \lambda^2 A_N + \cdots$$

$$+ \left[\frac{(\rho^{N-2}-\rho^{N-1})(1-2\beta^{-1})}{1-\rho^{N-1}}\right] \lambda^{N-1} A_N + (\rho\lambda)^{N-1} A_N ,$$

and further reduces to the fundamental equation

$$\sum_{j=2}^{N-1} \lambda^j \left[\frac{(\rho^{j-1}-\rho^j)(1-2\beta^{-1})}{1-\rho^j}\right] + \lambda + (\rho\lambda)^{N-1} - 1 = 0 . \quad (2.5)$$

Equation 2.5 defines the admissable region for α,β pairs for selected basin shape and size. At this point it is instructive to note that if there was no jumping and no overland flow directly into streams other than first order (or, if the overland flow in adjacent basins was equal in area drained), then the basin area ratio would be unity. However, overland flow directly into streams of all orders is always present and is geometrically related to the stream length ratio. Thus, although λ is a measure of both overland flow and the process of jumping, once the order of the parent basin is specified and the stream length ratio determined,

the overland flow is fixed. Consequently, since only P_i, as given by equation 2.1, can vary, the bounds on P_i determine the bounds on λ. Hence, λ assumes its minimum value when P_i is a maximum, i.e., $P_i = (\beta-2)/\beta$, all $i<N$, and λ assumes its maximum value when $P_i = 0$, all $i<N$. Therefore, equation 2.5 can be solved for three values of P_i. With $P_i = 0$, all $i<N$, we obtain

$$\lambda + (\rho\lambda)^{N-1} - 1 = 0, \tag{2.6}$$

which defines the upper bound on the admissable region for the α,β pairs. With $P_i = (\beta-2)/\beta$, all $i<N$, we obtain

$$\left(\frac{\beta-2}{\beta}\right) \sum_{j=2}^{N-1} \lambda^j \left[\frac{1-\rho^{-1}}{1-\rho^{1-j}}\right] + \lambda + (\rho\lambda)^{N-1} - 1 = 0, \tag{2.7}$$

which defines the lower bound on the admissable region for the α,β pairs. With P_i defined as indicated by equation 2.1 we obtain equation 2.5 which defines the most probable α,β values admissable for each N. Now, by using equation 2.4 to identify the interdependence of the four parameters we can write equations 2.5, 2.6, and 2.7, respectively, as

$$\sum_{j=2}^{N-1} (\beta\alpha^{-1/h})^j \left[\frac{(\rho^{j-1}-\rho^j)(1-2\beta^{-1})}{1-\rho^j}\right] + \beta\alpha^{-1/h} + \alpha^{(1/h)(h-1)(N-1)} - 1 = 0, \tag{2.8}$$

$$\beta - \alpha^{1/h} + \alpha^{(1/h)[(h-1)(N-1)+1]} = 0 , \tag{2.9}$$

and

$$\left(\frac{\beta-2}{\beta}\right) \sum_{j=2}^{N-1} (\beta\alpha^{-1/h})^{j} \left[\frac{1-\rho^{-1}}{1-\rho^{1-j}}\right] + \beta\alpha^{-1/h} + \alpha^{(1/h)(h-1)(N-1)} - 1 = 0 \quad (2.10)$$

Solution of equations 2.8, 2.9, and 2.10 yield the admissable region for the α,β pairs for each parent basin and specified h. The equations can be solved using such techniques as the Newton-Raphson method of successive approximation [Ref. 3]. Figure 2-2 illustrates the solutions for N = 8 and h = 0.55, 0.6, and 0.65. Table 2-2 reflects selected values for the solution of equation 2.8, the equation representing the most probable state. The solution of equation 2.8 using h = 0.6 will be referred to hereinafter as the equilibrium solution as it represents the most probable state within any Nth-order basin.

TABLE 2-2

EQUILIBRIUM SOLUTION OF FUNDAMENTAL EQUATION

N = 8					
h = 0.65		h = 0.6		h = 0.55	
α	β	α	β	α	β
1.72	2.00	1.62	2.00	1.54	2.00
2.00	2.38	2.00	2.61	2.00	2.89
2.50	3.18	2.50	3.61	2.50	4.18
3.00	4.11	3.00	4.80	3.00	5.78
3.50	5.15	3.50	6.19	3.50	7.68
4.00	6.31	4.00	7.75	4.00	9.87

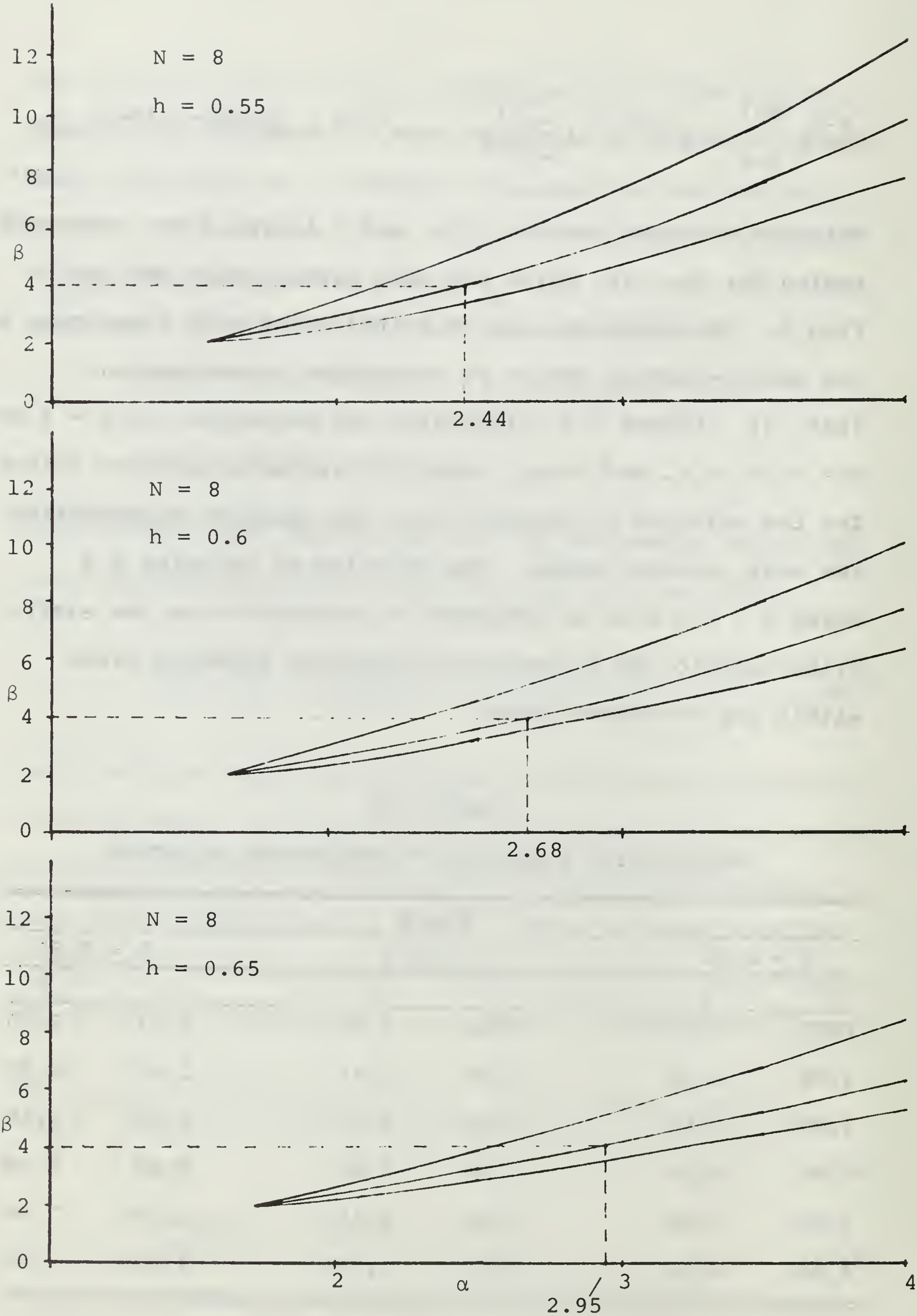


Figure 2-2. Admissable Region for α,β Pairs

V. A PARAMETRIC EVALUATION

A. THE AREA EQUATION

Observing that the total area drained by an Nth-order stream system consists solely of area drained by overland flow occurring directly into streams of order $i (i = 1, 2, ---, N)$, we can write the area of the parent basin as

$$A_N = n_1\bar{a}_1 + n_2\bar{a}_1\alpha + \ldots + n_N\bar{a}_1\alpha^{N-1}$$

$$= \bar{a}_1 \sum_{j=1}^{N} n_j\alpha^{j-1}$$

$$= \bar{a}_1 \sum_{j=1}^{N} \beta^{N-j}\alpha^{j-1} ,$$

which sums to

$$A_N = \bar{a}_1 \cdot \frac{\alpha^N - \beta^N}{\alpha - \beta} , \tag{3.1}$$

and further converts to the fundamental Hack equation [Ref. 27]

$$A_N = \bar{a}_1\beta^{N-1} \cdot \frac{\rho^N - 1}{\rho - 1} . \tag{3.2}$$

Using the latter equation and forming the basin area ratio we obtain

$$\frac{A_{N-1}}{A_N} = \frac{\bar{a}_1 \beta^{N-2} [\frac{\rho^{N-1}-1}{\rho-1}]}{\bar{a}_1 \beta^{N-1} [\frac{\rho^N - 1}{\rho-1}]} = \beta^{-1} \; \frac{\rho^{N-1}-1}{\rho^N-1} \; .$$

However, the basin area ratio as defined by equation 1.4 requires that all areas A_{N-1} in size within the parent basin be considered as the numerator and not just one such area as described by the Hack equation. Thus, the basin area ratio becomes

$$\lambda = \frac{\beta A_{N-1}}{A_N} = \frac{\rho^{N-1}-1}{\rho^N - 1} \; , \qquad (3.3)$$

which defines the upper bound on λ since Hack did not envision the process of jumping in his development. This curious omission of Hack's plus the fact that his development does consider overland flow occurring directly into streams of order greater than one implies that ratio 3.3 differs from unity by the fraction of overland flow occurring directly into the Nth-order stream, i.e.,

$$\lambda = 1 - \frac{F_N}{A_N} \; .$$

It can easily be demonstrated that F_N/A_N approaches zero rapidly as N increases. Thus, equation 3.3 is essentially equivalent to equation 2.5 with $P_i = 0$, all $i<N$, the latter implying that

$$\lambda + (\lambda\rho)^{N-1} = 1 \;. \qquad (3.4)$$

The closeness of equations 3.3 and 3.4 is illustrated by the results contained in Table 3-1 which were achieved using $N = 8$, α-values of 2, 3, and 4, and upper bound values on β (Figure 2-2) for each α value.

TABLE 3-1

COMPARATIVE VALUES FOR λ USING EQUATIONS 3.3 AND 3.4

N	Equation 3.3 (λ)		Equation 3.4 (λ)	
	ρ=.656	ρ=.398	ρ=.656	ρ=.398
1	0	0	0	0
2	0.60	0.72	0.60	0.72
3	0.75	0.88	0.79	0.90
4	0.84	0.95	0.88	0.96
5	0.89	0.98	0.93	0.98
6	0.92	0.99	0.95	0.99
7	0.94	1.00	0.97	1.00
8	0.97	1.00	0.98	1.00
9	0.98	1.00	0.99	1.00
10	0.99	1.00	0.99	1.00

Since the two equations are essentially equivalent, we may interchange them in equation 2.5 to obtain

$$\sum_{j=2}^{N-1} \left[\frac{\rho^{N-1}-1}{\rho^{N}-1}\right]^{j} \left[\frac{(\rho^{j-1}-\rho^{j})(1-2\beta^{-1})}{1-\rho^{j}}\right] + \frac{\rho^{N-1}-1}{\rho^{N}-1} + \left[\frac{\rho(\rho^{N-1}-1)}{\rho^{N}-1}\right]^{N-1} - 1 = 0 \tag{3.5}$$

Thus, we have demonstrated the relationship between the Hack equation (3.1) and equation 2.5, the fundamental equation of this thesis.

B. THE COEFFICIENT OF DRAINAGE DENSITY

The geometric laws of Horton, Schumm, and Hack from which were deduced the parameters α, λ and ρ are based on the values of $\bar{\ell}_1$ and $\bar{a}_1$ which for simplicity of development most authors assume to be unity. In this model, however, we are concerned as much with absolute values as we are with the constancy of their adjacent order ratios. Thus, it is necessary that we examine the behavior of $\bar{a}_1$ and $\bar{\ell}_1$ within relatively homogeneous drainage basins. To do this we define D to be the drainage density within a basin area measured in linear units of stream length per square unit of basin area (usually in miles per square mile). The density of streams within an area is, of course, a measure of how well the area is drained. A well-drained area would have a relatively low D-value and, conversely, a poorly drained area would have a relatively high D-value. The equation for D is

$$D = \frac{1}{A_N} \sum_{i=1}^{N} \bar{\ell}_i n_i ,$$

which, using equation 1.5, 1.6, and 3.1, sums to

$$D = \bar{\ell}_1/\bar{a}_1 \ . \tag{3.6}$$

Thus, we see that D is constant within any basin within which the aforementioned laws hold. Now, the relationship between $\bar{a}_1$ and $\bar{\ell}_1$ within a basin area is defined by equation 2.3. The behavior of this equation for the equilibrium values of $k = 1.4$ and $h = 0.6$ is illustrated in Figure 3-1. Note that for a value of $\bar{a}_1$ of 0.3 square miles, $\bar{\ell}_1$ is 0.7 miles and D is 2.27 miles per square mile.

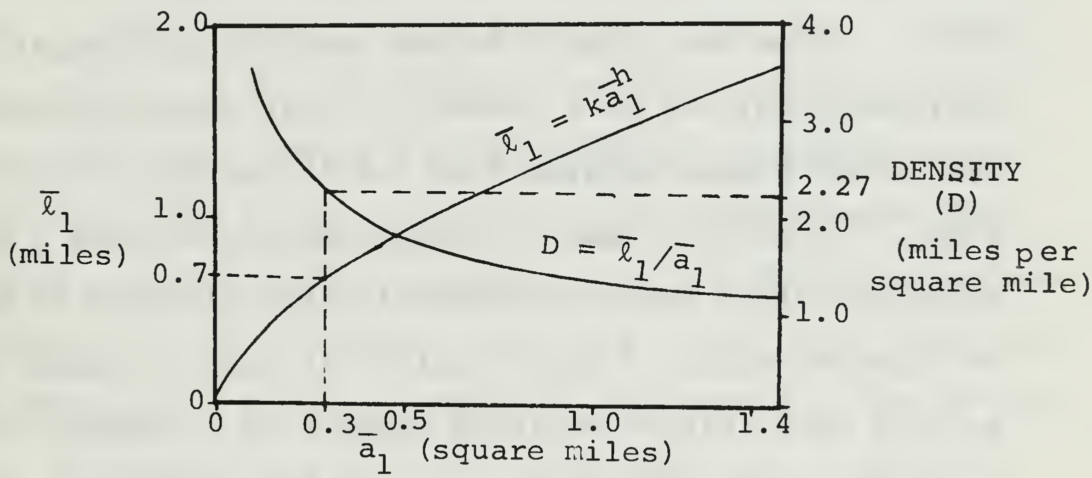


Figure 3-1. Drainage Density Versus Length and Area of the First Order Basins

Thus, we see that the drainage density is constant and uniquely determined for any given pair of values for $\bar{\ell}_1$ and $\bar{a}_1$. In an attempt to discover some reasonable values for

these constants we refer to a number of empirical investigations. In a study of 13,376 first order streams in the Appalachian Plateau, Morisawa [Ref. 33] found that the mean stream length was 0.083 miles, the mean basin area was 0.034 square miles, and the mean drainage density was 2.44 miles per square mile. This corresponds closely to the equilibrium solution illustrated by Figure 3-1. In a study of 146 first order streams along the upper Hiwassee River, New York, Horton [Ref. 11] found that a drainage density of 2.06 was prevalent. However, in a study of a fifth order stream system at Perth Amboy, New Jersey, Schumm [Ref. 34] found that the drainage density was 602 miles per square mile. In the same report Schumm recorded and compared drainage densities in a number of other areas throughout the United States ranging form 4.6 to the 602 in Perth Amboy. Obviously, then, drainage density assumes a wide range of values between different areas although it may be constant within a given basin. In fact, a number of authors have related drainage density to a texture ratio, the latter defined to be a measure of the closeness or proximity of one channel to another. In this respect, a book by Leopold, Wolman, and Miller [Ref. 1] contains the relationship redrawn as Figure 3-2. Thus, we conclude that the area (A_N) and main stream length (ℓ_N) of a parent basin of any given order is directly related to the drainage density and as a result varies substantially from one locale to another. Hence, any attempt to develop basin

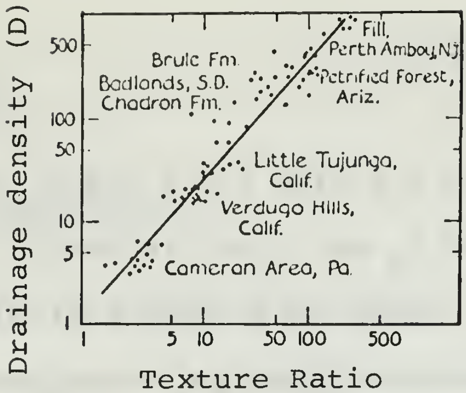


Figure 3-2. Relation of Drainage Density to Texture Ratio, which is Obtained by Dividing the Number of Contour Crenulations on the most Crenulated Contour by the Perimeter of the Basin. [Extracted from Leopold, Wolman, and Miller [Ref.1]]

areas or main stream lengths using as a base $\bar{a}_1$ = 1 square unit and $\bar{\ell}_1$ = 1 unit is likely to produce results substantially in error. Thus, the mathematical structure of our model will have to be a function of the actual basin area. However, as indicated by equation 3.6, the drainage density appears to remain constant within any relatively homogeneous drainage area. Now, in Chapter II we assumed in the development of equation 2.4 that the coefficient (k) in equation 2.3 was relatively constant within basins of adjacent orders. We will now demonstrate that the constancy of this coefficient is largely established by the constancy of the drainage (D). To do this we choose α,β pairs representing the equilibrium solution within the α,β admissable region. By initially setting $\bar{a}_1 = \bar{\ell}_1 = 1$ and solving equation 3.1 together with the relationship

$$\ell_N = \bar{\ell}_1 \alpha^{N-1} , \tag{3.7}$$

which is equation 1.6 with i = N and $\bar{\ell}_i = \ell_N$, we obtain the values for ℓ_8 and A_8 set forth in Table 3-2. We are able to let $\bar{a}_1$ and $\bar{\ell}_1$ equal unity here initially since we are interested in demonstrating the dependency of k on the drainage density (D) and not in its absolute values. Consequently, the values for ℓ_8 and A_8 set forth in Table 3-2 have no absolute significance.

TABLE 3-2

LENGTH AND AREAS OF AN EIGHTH ORDER BASIN FOR VARIOUS EQUILIBRIUM α,β PAIRS

α	β	ℓ_8	A_8
2.0	1.96	64	422
2.5	3.01	244	3193
3.0	4.24	729	18104
3.5	5.65	1838	82495
4.0	7.24	4096	316772

Now, by varying the drainage density we can calculate various sets of ℓ_8, A_8 - values in the manner used to produce those set forth in Table 3-2. We then perform a least-squares regression analysis on each set to examine the behavior of k and h in the power equation

$$\ell_8 = k\, A_8^h \,.$$

The regression equations are

$$\Sigma \log A_8 = T\, \Sigma \log \ell_8 + m \log b,$$

$$\Sigma (\log A_8 \log \ell_8) = T\, \Sigma (\log \ell_8)^2 + \log b\, \Sigma \log \ell_8,$$

and

$$\ell_8 = \left(\frac{1}{b} A_8\right)^{\frac{1}{T}} .$$

The analyses give the results set forth in Table 3-3.

TABLE 3-3

RESULTS OF LEAST-SQUARES REGRESSION ANALYSES FOR VARIOUS DRAINAGE DENSITIES

D	By Changing $\bar{\ell}_1$		By Changing $\bar{a}_1$	
	k	h	k	h
0.25	0.37	0.63	0.62	0.63
0.50	0.75	0.63	0.96	0.63
0.75	1.12	0.63	1.24	0.63
1.00	1.49	0.63	1.49	0.63
1.50	2.24	0.63	1.92	0.63
2.00	2.98	0.63	2.30	0.63

Thus, we see as did Hack [Ref. 27] that the coefficient (k) is a function of the drainage density (D) and that the exponent (h) is necessarily a function of the overland flow ratio (ρ). Actually, as pointed out by Hack, k is also affected by the value of ρ but to a very small extent. Now, since k is largely a function of drainage density we would expect k to remain essentially constant within a relatively homogeneous basin. In this respect, a number of authors have found that not only does k remain relatively constant within any given basin, but actually tends toward a common value of 1.4. In a study of streams in the Shenandoah Valley, Virginia, Hack [Ref. 27] recorded the relationship depicted by Figure 3-3.

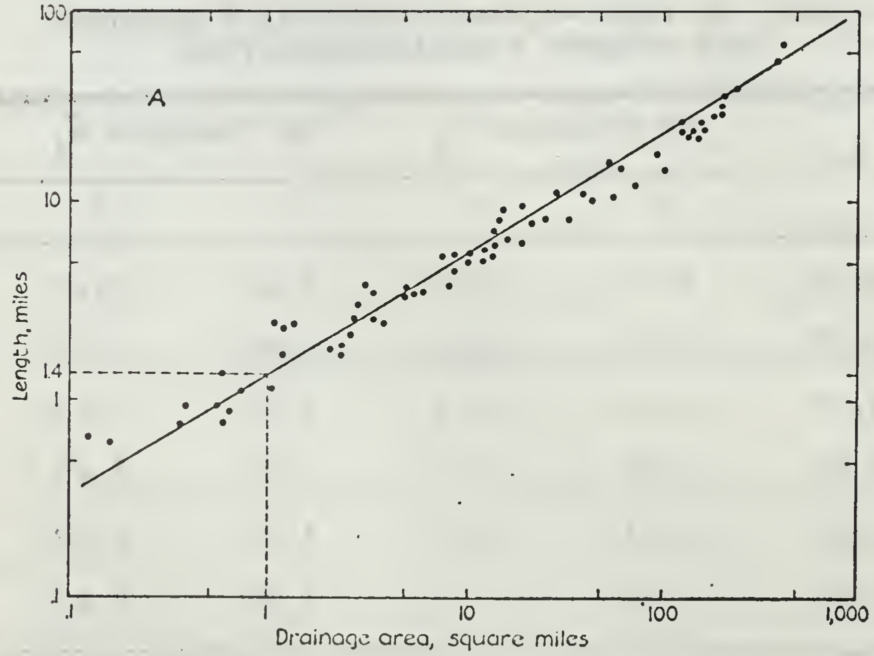


Figure 3-3. Relation of Channel Length in Miles to Drainage Area in Square Miles (redrawn from Hack [Ref. 27])

The relationship illustrated by Figure 3-3 has been verified by many authors in describing a number of widely scattered areas of the world. Thus, throughout the balance of this thesis we will refer to 1.4 as the equilibrium value of k.

C. THE BASIN SHAPE PARAMETER

Schumm confirmed the findings of most of the earlier authors in the field when he wrote "as the relief ratio increases the drainage basin becomes more elongate" [Ref. 34: 23]. In describing the drainage network of the badlands at Perth Amboy, New Jersey, Schumm found that as drainage systems mature and approach an equilibrium state, the relief increases and causes the basin shapes to become more elongate. Since we can characterize a drainage basin in its earliest stages of development as one in which no jumping has occurred, Schumm's findings imply that the process of maturing is closely allied with the growth of the jumping process. Thus, if we consider an Nth-order drainage basin in which jumping is prohibited, we would expect the basin shape parameter (h) to decrease toward 0.5 as the basin order decreases. The latter would be true since we have shown that h is unaffected by changes in the drainage density and since the jumping prohibition implies that all streams lie within the basins of the next higher order; the latter condition implying that both α and β remain constant within the included basins of orders less than N. Thus, we surmise that as stream networks are in their initial stages of development, the typical first order stream basin is

circular. As the network matures and larger streams are formed, the P_i can no longer equal zero for all $i<N$ and the basins begin to elongate in a downstream direction. This process continues until the network achieves a steady state and the basin shapes become relatively constant within the parent basin. This process can be illustrated by letting the elongation occur as a one-sided ellipse in the downstream direction. The circular area and the one-sided elliptical area are illustrated in Figure 3-4.

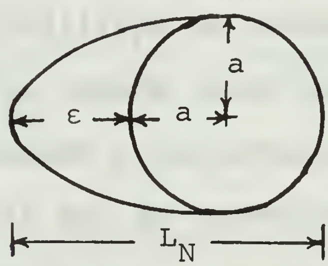


Figure 3-4. Typical Drainage Basin

The area-length relationship of the typical basin can be expressed as

$$A_N = \frac{\pi}{4} L_N (L_N - \varepsilon) \quad , \tag{3.8}$$

which, with $\varepsilon = 0$, is the equation of a circle. We can also write

$$L_N^{1/h} = L_N (L_N - \varepsilon) \quad ,$$

where

$$h = \frac{\log_e L_N}{\log_e L_N + \log_e (L_N - \varepsilon)} . \qquad (3.9)$$

Thus, relationship 3.8 becomes

$$L_N = (\frac{4}{\pi} A_N)^h . \qquad (3.10)$$

For a given length, h is a function of ε, the downstream elongation. As ε increases from zero, h increases from 0.5 towards unity but can never achieve unity since $\varepsilon < L_N$. Mayer [Ref. 32] and Hack [Ref. 27] found that basin length can be more closely represented by

$$L_N = A_N^h , \qquad (3.11)$$

which implies that the typical mature basin area is somewhat greater for a given length than is portrayed by the one-sided ellipse. In any event, the area of the drainage basin does vary as the hth-root of the basin length, the value of h being a function of the overland flow ratio (ρ). Solving equation 3.11 for h and substituting therein equations 3.1 and 3.7 we can obtain the equation for h for the general case i. Then forming the ratio of adjacent basin shape parameters we further obtain

$$\frac{h_{i-1}}{h_i} = \frac{\log_e k\bar{\ell}_1 \alpha^{i-2}}{\log_e k\bar{\ell}_1 \alpha^{i-1}} \cdot \frac{\log_e [\bar{a}_1 \cdot \frac{\alpha^i - \beta^i}{\alpha - \beta}]}{\log_e [\bar{a}_1 \cdot \frac{\alpha^{i-1} - \beta^{i-1}}{\alpha - \beta}]} , \qquad (3.12)$$

where k is the constant that converts the length of the main stream given by equation 2.3 into the length of the basin given by equation 3.11, i.e., the main stream is k times as long as the basin. If we now consider a basin which has achieved equilibrium with its environment, we can see that, although α is unaffected, β decreases within included basins of successively lower order since streams that jump do not lie within the basins of the next higher order. We can describe the behavior of the bifurcation ratio within the included basins by letting β_i represent the bifurcation ratio within the ith-order basin and observing that

$$\beta_i = \beta q_{i-1} ,$$

where β is the constant bifurcation ratio for the parent basin and q_{i-1} the proportion of (i-1)st-order streams that don't jump. Using equation 2.1 to obtain q_{i-1} we can write

$$\beta_i = \frac{(1-\rho) + 2\beta^{-1}(\rho-\rho^{N-i+1})}{1-\rho^{N-i+1}} \cdot \beta \qquad (3.13)$$

Now, if h is relatively constant for adjacent included basins as assumed in Chapter II, equation 3.12 should give solutions near unity as i varies when α remains constant and β_i is defined by equation 3.13. Solution of the equation for N = 8 and the equilibrium values of $\alpha = 2.68$, $\beta = 4$, and k = 1.4 disclose values of .99+ as N increases from 3 to 8 indicating that the assumption is valid for included basins of order near N. Since equation 2.4 requires only

that h be constant for the Nth- and (N-1)st-order basins, we see that the requirement is fully met. This theoretical development is supported by a number of empirical investigations. The specific conclusion concerning the constancy of h has been substantiated by Hack who, in commenting on his equation $L = bA^{n}$, stated

> the exponent n must always have approximately the same value in any homogeneous area for its value is determined by the laws of probability and not by geologic factors [Ref. 26:B17].

In examining streams in Michigan, Virginia, and Maryland, Hack [Ref. 27] further found that as the drainage basins became more homogeneous, h approached a value of 0.6 and k a value of 1.4. In their random walk model of a hypothetical drainage system, Leopold and Langbein [Ref. 28] found that a value of 0.64 held for h. This latter finding was significant not only for the value of h but because it strongly supported Hack's finding that h was the product of chance factors. In a Horton analysis of 47 small watersheds in Illinois, Iowa, Missouri, Nebraska, North Carolina, Ohio, and Wisconsin, Gray [Ref. 9] found that values of 1.4 and 0.568 held for k and h, respectively. Schenck [Ref. 18] used a Monte-Carlo random walk computer simulation to show, among other things, that Gray's findings were substantiated by considerations of the most probable state. One of the more significant substantiations was contributed by Leopold, et al., [Ref. 1] by providing a plot of many of the world's major rivers showing an adherence to

a 0.6 value for h worldwide. Mayer [Ref. 32] observed that a constant h does not imply a constant length-to-width ratio and provided the values set forth in Table 3-4 as an illustration.

TABLE 3-4

LENGTH-TO-WIDTH RATIOS OF DRAINAGE BASINS
(REDRAWN FROM MAYER [REF. 32])

h = 0.6			
Area (sq mi)	Length (mi)	Width (mi)	Length/ Width
100,000	1,000	100	10.0
10,000	250	40	6.2
1,000	63	16	3.9
100	16	6.3	2.5
10	4	2.5	1.6
1	1	1.0	1.0

C. MEAN BASIN WIDTH

Hack [Ref. 27] also observed that the width of a typical basin could be approximated by the mean width ($\overline{W}$) which, in effect, reduces the basin shape to a rectangle. Thus, we can write

$$A_N = L_N\overline{W}_N ,$$

which substituted into equation 3.11 yields

$$\overline{W}_N = A_N^{1-h} \quad . \tag{3.14}$$

Using the mean width would most likely yield unsatisfactory results if only one or only a few drainage basins were crossed. It would, of course, depend on where the basins were crossed. However, if a sufficiently large number of basins is considered and the points of crossing randomly determined, the expected crossing point would occur where the basin assumes its mean width. Thus, in this context, which is quite reasonable from a planning viewpoint if the standard deviation is not too great, the mean width is fully acceptable.

VI. A CROSSING ALGORITHM

A. THE HORIZONITAL COMPONENT OF STREAM LENGTH

We consider an Nth-order parent basin possessing mean characteristics in which $\bar{\ell}_i$ and n_i again denote, respectively, the mean length of the ith-order streams and the number of ith-order streams within the mean basin area. We assume that the streams are randomly distributed and randomly oriented within the drainage basin and that θ, as given by Figure 4-1, is uniform on the interval (0, π/2).

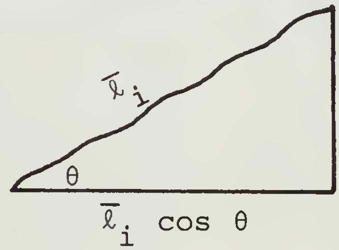


Figure 4-1. The Horizonital Component of Mean Stream Length

We further envision a force of width ω traversing the mean basin parallel to the minor axis and define the horizontal component of each mean stream length as that length perpendicular to the axis of traverse. The horizonital component is given by $\bar{\ell}_i \cos\theta$ in Figure 4-1. The density function of θ can be written as

$$f(\theta) = \begin{cases} 2/\pi \ , & 0 < \theta < \pi/2 \\ 0 \ , & \text{elsewhere} \end{cases} \quad . \tag{4.1}$$

The expected length of the horizonital component can be shown to be

$$E[\bar{\ell}_i \cos\theta] = \int_0^{\pi/2} |\bar{\ell}_i \cos\theta|\, f(\theta)\, d\theta = \frac{2}{\pi}\,\bar{\ell}_i \qquad (4.2)$$

B. THE STREAM PROBABILITY DENSITY FUNCTION

We denote the expected length of the horizonital component as $\hat{\ell}_i$ and consider the midpoint of each. Using the crossing force of width ω we define the relationship portrayed by Figure 4-2, where the coordinate system is envisioned to be superimposed over the mean basin area.

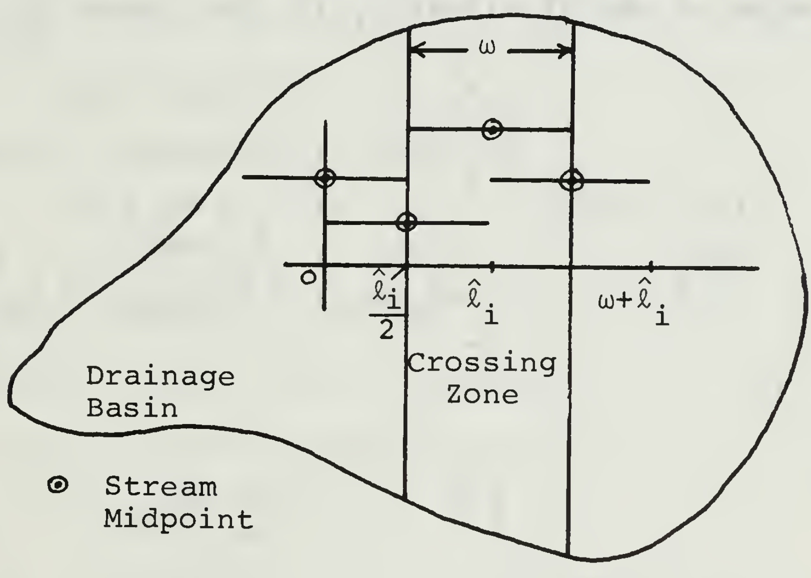


Figure 4-2. The Relationship Between Crossing Width and Midpoint of Horizonital Component. A midpoint to the left of zero and to the right of $\omega+\hat{\ell}_i$ implies that the respective stream is not crossed. The midpoint location within the interval zero to $\omega+\hat{\ell}_i$, inclusive, then, describes the length of stream protruding into the crossing zone given that the stream is crossed

Since streams are randomly distributed within the mean basin, the midpoints of $\hat{\ell}_i$ are uniform on the interval $(0,\omega+\hat{\ell}_i)$. We denote the midpoint of an $\hat{\ell}_i$ as X_i and define it as the position, measured from the origin, of the midpoint of the ith-order stream. We also define Y_i as the extension into the crossing zone of the ith-order stream. We then let x_i and y_i represent the actual points and lengths, respectively, and define

$$\delta_1 = \text{set of indices } i \text{ such that } \hat{\ell}_i \le \omega,$$

and $$\delta_2 = \text{set of indices } i \text{ such that } \hat{\ell}_i > \omega.$$

The value of the statistic y_i is then given by

$$\underset{i\varepsilon\delta_1}{y_i} = h_1(x_i) = \begin{cases} x_i & , \ 0 \le x_i < \hat{\ell}_i \\ \hat{\ell}_i & , \ \hat{\ell}_i \le x_i \le \omega \\ \omega+\hat{\ell}_i-x_i & , \ \omega < x_i \le \omega + \hat{\ell}_i \end{cases} \quad , \tag{4.3}$$

and

$$\underset{i\varepsilon\delta_2}{y_i} = h_2(x_i) = \begin{cases} x_i & , \ 0 \le x_i < \omega \\ \omega & , \ \omega \le x_i \le \hat{\ell}_i \\ \omega+\hat{\ell}_i-x_i & , \ \hat{\ell}_i < x_i \le \hat{\ell}_i + \omega \end{cases} \quad . \tag{4.4}$$

The distribution of X_i, conditional on a stream being encountered, is given by

$$g(x_i) = \begin{cases} \dfrac{1}{\omega+\hat{\ell}_i} & , \quad 0 < x_i < \omega + \hat{\ell}_i \\ 0 & , \quad \text{elsewhere} \end{cases} . \tag{4.5}$$

The expected length of an ith-order stream extending into the crossing zone is then given by

$$\underset{i\varepsilon\delta_1}{E[y_i]} = \int_0^{\omega+\hat{\ell}_i} h_1(x_i)g(x_i)dx_i = \frac{\hat{\ell}_i\omega}{\omega + \hat{\ell}_i} , \tag{4.6}$$

and

$$\underset{i\varepsilon\delta_2}{E[y_i]} = \int_0^{\omega+\hat{\ell}_i} h_2(x_i)g(x_i)dx_i = \frac{\hat{\ell}_i\omega}{\omega + \hat{\ell}_i} . \tag{4.7}$$

The respective variances are given by

$$\underset{i\varepsilon\delta_1}{\sigma^2_{y_i}} = \frac{\hat{\ell}_i^3(2\omega-\hat{\ell}_i)}{3(\omega+\hat{\ell}_i)^2} , \tag{4.8}$$

and

$$\underset{i\varepsilon\delta_2}{\sigma^2_{y_i}} = \frac{\omega^3(2\hat{\ell}_i-\omega)}{3(\omega+\hat{\ell}_i)^2} . \tag{4.9}$$

The foregoing could perhaps be made clearer by considering the density of y_i given by

$$\underset{i\varepsilon\delta_1}{g_1(y_i)} = \begin{cases} \dfrac{2}{\omega+\hat{\ell}_i}, & 0 \le y_i < \hat{\ell}_i \\ \dfrac{\omega-\hat{\ell}_i}{\omega+\hat{\ell}_i}, & y_i = \hat{\ell}_i \end{cases}, \tag{4.10}$$

and

$$\underset{i\varepsilon\delta_2}{g_2(y_i)} = \begin{cases} \dfrac{2}{\omega+\hat{\ell}_i}, & 0 \le y_i < \omega \\ \dfrac{\hat{\ell}_i-\omega}{\omega+\hat{\ell}_i}, & y_i = \omega \end{cases}. \tag{4.11}$$

Using these mixed densities it can be easily verified that y_i, defined over the two regions indicated, does, in fact, have the mean and variances indicated by equations 4.6 through 4.9.

C. CONSTRUCTION OF THE MODEL

We now define η_i as the number of ith-order streams extending into the crossing zone. The expected total length of ith-order streams extending into the crossing zone is then given by

$$\Phi_i = E[\eta_i y_i] = \frac{\eta_i \hat{\ell}_i \omega}{\omega + \hat{\ell}_i} .$$

The behavior of stream lengths extending into the crossing zone as a function of midpoint position is illustrated by Figure 4-3.

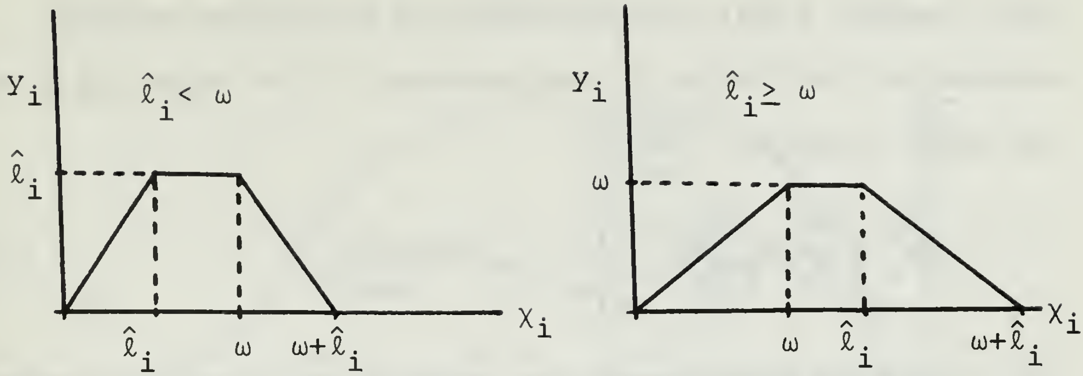


Figure 4-3. Behavior of Stream Lengths in Crossing Zone

Since, by assumption, the midpoints of streams are uniformly distributed within the mean basin area, we can consider that the actual density of midpoints per unit area is given by

$$\frac{n_i}{A_N}, \; i = 1, 2, ---, N.$$

Now, if a midpoint is within the width $\omega + \hat{\ell}_i$, the corresponding stream extends into the crossing zone. Thus, the midpoint area traversed by the crossing force in crossing the mean basin along the minor axis is given by

$$\overline{W}_N(\omega + \hat{\ell}_i),$$

and the number of ith-order streams extending into the crossing zone is, therefore,

$$\eta_i^* = \frac{n_i}{A_N} \overline{W}_N(\omega + \hat{\ell}_i) \quad . \tag{4.12}$$

The expected total stream length of ith-order streams crossed by the force in one traverse of the mean basin along the minor axis is, then,

$$\Phi_i^* = \frac{n_i}{A_N} \overline{W}_N \hat{\ell}_i \omega \quad . \tag{4.13}$$

The foregoing development can perhaps be better motivated by considering the following argument. Assume the following as we have previously:

1. The stream midpoints are uniformly distributed within the mean basin area.

2. The crossing zone is randomly chosen in the mean basin area.

3. In any portion of the crossing zone the crossing force encounters any stream having its midpoint within the width $\omega + \hat{\ell}_i$ and does not encounter any stream not having its midpoint within this width.

Now divide the crossing zone into n equal portions indicated in Figure 4-4. If n is large enough so that most of the small areas are randomly related to any particular one, the chance of failing to encounter a particular stream during the traverse is the product of the chances that encounter fails during motion along each small piece. Now the probability that the stream is encountered in a small area is the probability that the stream is in the small area which is given by

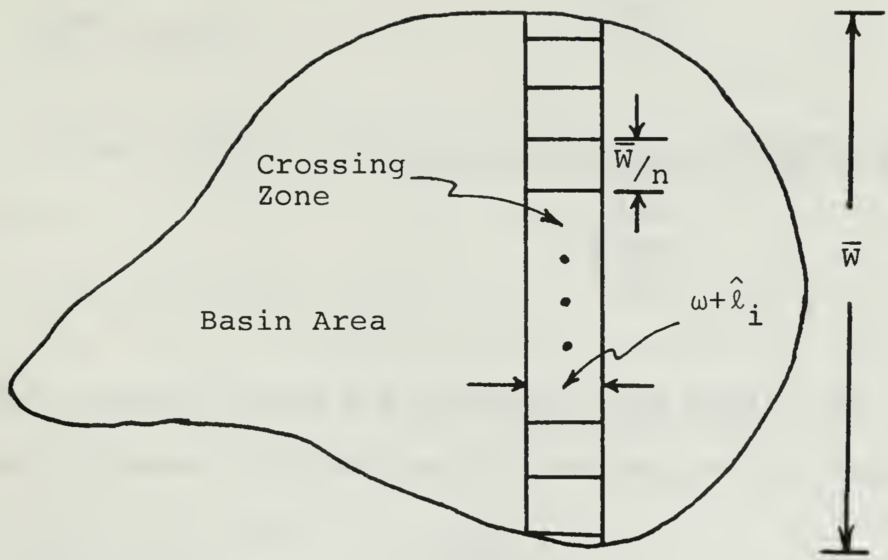


Figure 4-4. Division of Crossing Zone Into Arbitrarily Small Areas

$$\frac{\overline{W}}{nA_N}\,(\omega + \hat{\ell}_i)\ .$$

Hence, the chance that along the entire traverse the stream is not encountered is

$$\left[1 - \frac{\overline{W}}{nA_N}\,(\omega + \hat{\ell}_i)\right]^n .$$

Thus, as n gets large the probability of encountering the stream approaches

$$1 - e^{-\frac{\overline{W}(\omega+\hat{\ell}_i)}{A_N}}\ .$$

This further converts for minor axis traverse to

$$1 - e^{-\frac{(\omega+\hat{\ell}_i)}{A_N^h}} ,$$

and for major axis traverse to

$$1 - e^{-\frac{(\omega+\hat{\ell}_i)}{A_N^{1-h}}} .$$

Now, the latter two functions are both cumulative distributions for the exponential probability density functions

$$A_N^{-h} (\omega+\hat{\ell}_i) e^{-\frac{(\omega+\hat{\ell}_i)}{A_N^h}} ,$$

and

$$A_N^{h-1} (\omega+\hat{\ell}_i) e^{-\frac{(\omega+\hat{\ell}_i)}{A_N^{1-h}}} .$$

Since the distance between encounters is exponential with mean

$$\frac{A_N}{\omega+\hat{\ell}_i} ,$$

the underlying process must be Poisson with mean for a minor axis traverse of

$$A_N^{-h} (\omega+\hat{\ell}_i) ,$$

and for a major axis traverse of

$$A_N^{h-1}\ (\omega+\hat{\ell}_i)\ .$$

Hence, we can write the expected number of streams encountered as

$$\eta_i^* = A_N^{-h}\ \beta^{N-i}\ (\omega+\hat{\ell}_i)\ ,$$

and

$$\eta_i^{**} = A_N^{h-1}\ \beta^{N-i}\ (\omega+\hat{\ell}_i)\ ,$$

which, for the minor axis traverse, is precisely equation 4.12. This development does not give us expected stream length in crossing zone but does give us the added feature of mean interfluvial distance, i.e., distance between streams, which, using the mean of the exponential distribution, can be expressed as

$$d_i = \frac{A_N}{\beta^{N-i}(\omega+\hat{\ell}_i)}\ . \tag{4.14}$$

Now, from equation 1.1 we can deduce that

$$\bar{\ell}_i = \ell_N \alpha^{i-N}\ .$$

However, this equation reflects Horton stream lengths which are unsuitable for our purposes. Converting them to Strahler stream lengths using equation 1.9, we get

$$\overline{\ell}_i = \ell_N \, (\alpha^{i-N} - \alpha^{i-N-1}), \; i > 1 \; ,$$

which, using equations 2.3 and 3.11, further converts to

$$\overline{\ell}_i = k \, A_N^h \, (\alpha^{i-N} - \alpha^{i-N-1}), \; i > 1. \qquad (4.15)$$

Using the fact that $\hat{\ell}_i = \frac{2}{\pi} \overline{\ell}_i$ along with equations 1.5, 3.11, 3.14, and 4.15 we can express equations 4.8, 4.9, 4.12, 4.13, and 4.14 in more tractable terms as

$$\sigma^2_{\substack{y_i \\ i\varepsilon\delta_1}} = \frac{[\frac{2k}{\pi} A_N^h(\alpha^{i-N} - \alpha^{i-N-1})]^3 \, [2\omega - \frac{2k}{\pi} A_N^h(\alpha^{i-N} - \alpha^{i-N-1})]}{3[\omega + \frac{2k}{\pi} A_N^h(\alpha^{i-N} - \alpha^{i-N-1})]^2}, i>1, \qquad (4.16)$$

$$\sigma^2_{\substack{y_i \\ i\varepsilon\delta_2}} = \frac{\omega^3 [\frac{4k}{\pi} A_N^h \, (\alpha^{i-N} - \alpha^{i-N-1}) - \omega]}{3[\omega + \frac{2k}{\pi} A_N^h \, (\alpha^{i-N} - \alpha^{i-N-1})]^2} \; , \; i > 1, \qquad (4.17)$$

$$\eta_i^* = \beta^{N-i} \, [\omega A_N^{-h} + \frac{2k}{\pi} \, (\alpha^{i-N} - \alpha^{i-N-1})], \; i > 1, \qquad (4.18)$$

$$\Phi_i^* = \beta^{N-i} \, \frac{2k}{\pi} \, \omega \, (\alpha^{i-N} - \alpha^{i-N-1}), \; i > 1, \qquad (4.19)$$

and

$$d_i = \beta^{i-N} \, A_N \, [\omega + \frac{2k}{\pi} A_N^h \, (\alpha^{i-N} - \alpha^{i-N-1})]^{-1} , \; i > 1 \; . \qquad (4.20)$$

Note: For $i = 1$ in the foregoing equations set $\alpha^{i-N-1} = 0$.

Equations 4.18 and 4.19 can also be made applicable to crossings of the mean basin along the major axis by substituting basin length for basin width. The equations then become

$$\eta_i^{**} = \beta^{N-i} A_N^{h-1} [\omega + \frac{2k}{\pi} A_N^{h} (\alpha^{i-N} - \alpha^{i-N-1})] , \quad (4.21)$$

and

$$\Phi_i^{**} = \beta^{N-i} \frac{2k}{\pi} \omega A_N^{2h-1} (\alpha^{i-N} - \alpha^{i-N-1}) . \quad (4.22)$$

Two additional aspects of the model of interest are the coefficient of variation (CV_i) and an upper bound (U_i) on the total expected stream length (Φ_i). The upper bound can be established by noting that the expected extension into the zone of one stream cannot be greater than ω or $\hat{\ell}_i$, whichever is appropriate, and the equations for the upper bound then written as

$$u_i = \eta_i \hat{\ell}_i , \quad i\varepsilon\delta_1 ,$$

and

$$u_i = \eta_i \omega , \quad i\varepsilon\delta_2 .$$

The coefficient of variation is the standard deviation divided by the mean and is given by the equations

$$CV_i = \sqrt{\frac{\hat{\ell}_i (2\omega - \hat{\ell}_i)}{3\omega^2}} , \quad i\varepsilon\delta_1 ,$$

and

$$CV_i = \sqrt{\frac{\omega(2\hat{\ell}_i - \omega)}{3\hat{\ell}_i^2}} \quad , \quad i\varepsilon\delta_2 \ .$$

D. MODEL APPLICATION

A number of methods are available to obtain values for the model parameters. If α, β, h, and k are known for the particular area in question then the model equations can be solved merely by measuring the area (A_N) of the basin to be crossed from a topographic map or aerial photograph. The area may be determined by using a planimeter or estimated using the equation $A_N = L_N W_N$, where W_N is taken as the width at the intended point of crossing. However, the latter method is understandably cruder than using a planimeter. In the event that either α or β is unknown, the other may be extracted from the appropriate curve in Figure 2-2, that is, by solving equation 2.8 for the appropriate value of h. In the event that none of the parameters is known, results for the most probable state may be obtained by using h = 0.6, k = 1.4, and β = 4, and again solving equation 2.8 for the appropriate α-value. Using β = 4 to represent the most probable state has been substantiated by a number of authors. Actually, β varies somewhat from basin to basin but clusters about 4 in mature basins. The sensitivity of the model with respect to the various parameters will be discussed in the next section. At this point two model applications will be examined.

1. <u>Application 1</u>. When planning for a combat force, such as a division, to move across a drainage area the conditions throughout the entire crossing zone could easily be of interest. To obtain this information we solve the model equations letting ω equal the actual width of the crossing force front. Solution of the model for a minor axis traverse for $\omega = 1$ mile, $h = 0.6$, $k = 1.4$, $\alpha = 2.68$, $\beta = 4$, $N = 8$, and $A_8 = 1500$ square miles is set forth in Table 4-1. The solution for a major axis traverse using the same parameter values is set forth in Table 4-2. In solving the equations of this model, it is important to note that $\hat{\ell}_i$ must be taken as min $[\hat{\ell}_i, \overline{W}-\omega]$ for a major axis traverse and min $[\hat{\ell}_i, L_N-\omega]$ for a minor axis traverse. This is necessary to preclude violation of the basin confines.

For discussion purposes consider the entry in Table 4-1 for a fourth-order stream. The model predicts that the eighth-order basin will contain 256 fourth-order streams all of which lie entirely within the confines of the basin area. If the crossing force has a crossing front of one mile ($\omega = 1$), the model predicts that approximately six of the streams will intersect the crossing zone and that the total expected length within the zone perpendicular to the crossing axis will be 2.77 miles. Now, given that a fourth-order stream intersects the crossing zone, the model further predicts that the expected extension into the zone of each stream is 0.47 miles, or about one-half the width

TABLE 4-1

MODEL SOLUTION FOR MINOR AXIS TRAVERSE OF EIGHTH-ORDER BASIN

Stream Order (i)	Number Streams in Basin (n_i)	Number Streams in Crossing Zone (η_i^*)	Total Horiz Length of Streams in Crossing Zone, miles (Φ_i^*)	Mean Horiz Stream Length in Crossing Zone, miles (y_i)	Standard Deviation of Stream Length, miles (σ_{y_i})	Upper Limit on Stream Length, miles	Coeff of Var (CV_i)	Mean Inter-fluvial Distance, miles (d_i)
1	16384	218.31	14.72	0.07	0.21	15.78	0.22	0.08
2	4096	57.08	6.18	0.11	0.18	6.93	0.28	0.33
3	1024	16.87	4.14	0.25	0.43	5.49	0.43	1.10
4	256	5.96	2.77	0.47	0.74	5.19	0.57	3.13
5	64	2.65	1.86	0.70	0.62	2.65	0.47	7.02
6	16	1.44	1.25	0.86	0.34	1.44	0.31	12.90
7	4	0.88	0.83	0.94	0.15	0.88	0.20	21.08
8	1	0.57	0.56	0.98	0.07	0.57	0.12	32.61

TABLE 4-2

MODEL SOLUTION FOR MAJOR AXIS TRAVERSE OF EIGHTH-ORDER BASIN

Stream Order (i)	Number Streams in Basin (n_i)	Number Streams in Crossing Zone (η_i^{**})	Total Horiz Length of Streams in Crossing Zone, miles (Φ_i^{**})	Mean Horiz Stream Length in Crossing Zone, miles (y_i)	Standard Deviation of Stream Length, miles (σ_{y_i})	Upper Limit on Stream Length, miles	Coeff of Var (CV_i)	Mean Inter-fluvial Distance, miles (d_i)
1	16384	942.53	63.54	0.07	0.92	68.13	0.22	0.08
2	4096	246.43	26.69	0.11	0.80	29.93	0.28	0.33
3	1024	72.82	17.88	0.25	1.87	23.70	0.43	1.10
4	256	25.71	11.98	0.47	3.20	22.43	0.57	3.13
5	64	11.46	8.03	0.70	2.66	11.46	0.47	7.02
6	16	6.24	5.38	0.86	1.45	6.24	0.31	12.90
7	4	3.82	3.60	0.94	0.67	3.82	0.20	21.08
8	1	1.00	0.95	0.95	0.17	1.00	0.19	32.61

of the crossing zone. The latter statistic implies that given sufficient maneuverability in zone it is conceivable that, if necessary, the fourth-order streams could be by-passed. A decreasing frontage ($\omega<1$) would serve to limit this possibility. The model then computes a standard deviation for the expected horizonital length of streams in zone of 0.74 miles and an upper bound of 5.19 miles. The upper bound is associated with certainty, i.e., given that six fourth-order streams intersect the crossing zone we can be 100 percent certain that the total extension into the zone of all fourth-order streams does not exceed 5.19 miles. For the particular probability density function used in this model (equations 4.10 and 4.11), the standard deviation for fourth-order streams is associated with approximately 54 percent confidence, i.e., we can be approximately 54 percent certain that the total extension into the crossing zone of all fourth-order streams lies between 2.03 and 3.51 miles. In fact, it can easily be shown that the maximum value attainable for the standard deviation is $\omega\Phi_i/3$ and that this maximum occurs when $i\varepsilon\delta_2$. An additional measure of the variation about the mean is provided by the coefficient of variation which is a dimensionless measure of the variation as a function of the mean. In general, the coefficient of variation for this model behaves in limiting form pursuant to the bounds

$$0 \leq CV \leq \frac{\sqrt{3}}{3}, \; i\varepsilon\delta_1 ,$$

$$0 \leq CV < \infty , \; i\varepsilon\delta_2 .$$

However, the upper limit on the latter set of bounds envisions streams of infinite length which, of course, is rather impractical. Consequently, the upper limit would behave in a more restricted fashion as reflected by the results in Tables 4-1 and 4-2. In fact, the magnitude of the coefficients indicate that the mean is a rather good approximation of the distribution. Finally, the model estimates the distance between successive fourth-order streams as 3.13 miles. Now, the number of parent streams in the crossing zone (η_8^*) has an additional and perhaps more enlightening interpretation. For the minor axis traverse of the eighth-order basin the number of parent streams is shown to be 0.57. This can be interpreted as the probability that the parent stream is crossed given a traverse at the point where the basin assumes its mean width. This is a particularly salient feature of the model since it is usually the main stream that possesses the greatest obstacle value.

2. Application 2. A combat force might be considered to move in one or more columns under circumstances that would require knowledge only of the streams intersecting the column(s), e.g., movement along a road or other confined route. This model accommodates these circumstances with

$\omega = 0$. Since length of streams in the crossing zone is of no concern under these circumstances only n_i, η_i, and d_i are applicable. Solutions using the parameter values of application 1 for both axes are set forth in Table 4-3.

TABLE 4-3

MODEL SOLUTION FOR EIGHTH-ORDER BASIN FOR TRAVERSE ALONG BOTH AXES WITH CONFINED CROSSING FRONTAGE

Stream Order (i)	No. Streams in Basin (n_i)	No. Streams in Crossing Zone		Mean Inter-fluvial Distance, miles (d_i)
		Minor Axis (η_i^*)	Major Axis (η_i^{**})	
1	16384	14.72	63.54	1.27
2	4096	6.18	26.69	3.01
3	1024	4.14	17.88	4.50
4	256	2.77	11.98	6.72
5	64	1.86	8.03	10.03
6	16	1.25	5.38	14.96
7	4	0.83	3.60	22.33
8	1	0.56	1.00	33.33

E. MODEL SENSITIVITY

Since the model describes equilibrium behavior, that is, the behavior of a drainage network that has reached maturity or a steady state, it is instructive to examine the sensitivity of the model results with respect to the four

parameters upon which the model is based. A five-percent perturbation of the parameters will be used, however, it should be understood that, in nature, a change in one of these parameters is accompanied by changes among the others to maintain the system in equilibrium. Hence, as pointed out by Mayer [Ref. 32], a five-percent change one at a time imposes a condition of unnatural stress for the system. Table 4-4 presents the perturbed results for the fundamental model output. The equilibrium (unperturbed) solution is the last entry in the table.

TABLE 4-4

MODEL SENSITIVITY FOR THIRD-ORDER STREAMS IN EIGHTH-ORDER BASIN, MINOR AXIS TRAVERSE

Variable Perturbed	Perturbed Variable Values	Results of Perturbation Third Order Streams		
		n_3	η_3^*	Φ_3^*
β	3.8	792	13.05	3.21
	4.2	1307	21.53	5.29
α	2.55	1024	17.87	5.15
	2.81	1024	16.08	3.36
h	0.57	1024	19.99	4.14
	0.63	1024	14.36	4.14
k	1.33	1024	16.66	3.93
	1.47	1024	17.07	4.35
Equilibrium Solution $\beta = 4$ $h = 0.60$ $\alpha = 2.68$ $k = 1.40$		1024	16.87	4.14

Obviously, the parameter to which the model is most sensitive is β as a five-percent change in β results in approximately a 22-percent change in the model output. Consequently, the model output would be far more reliable if the β-value used was, in fact, the actual β-value from the drainage basin being crossed. Another interesting feature of the sensitivity analysis is that although η_i^* is sensitive to h, Φ_i^* is not. This apparent anomaly is caused by the fact that the expected horizonital length (y_i) of the streams within the crossing zone varies in inverse proportion to the expected number (η_i^*) of streams in the crossing zone and cancels out the effect. Thus, Φ_i^* is not a function of basin shape, a fact that is apparent from equation 4.19. The model would, of course, produce more reliable results for a specific basin if all parameter values were determined from the basin being crossed. However, failing this, determining β and, perhaps, α for each basin would place the model results well within the realm of acceptability.

F. CONCLUSIONS

The model results should be viewed in full consideration of the assumptions underlying the model structure. A short review of those assumptions is instructive. In Chapter I we assumed that mean stream length, mean basin area, mean overland flow area, and numbers of streams are geometrically related to basin order, which, of course, is what makes the respective dimensionless ratios α, λ, ρ, and β constant. The validity of these geometric relationships is supported

by a great deal of empirical evidence and by considerations of the most probable state. However, the relationships were in large measure least squares approximations and some scatter did exist especially among the data from some of the less mature basins. We next assumed in Chapter II that streams bifurcate in direct proportion to the bifurcation opportunity and defined bifurcation opportunity as the ratio of the total length of streams of each of the higher orders to the total length of streams of all of the higher orders in the parent basin. We were thus able to completely describe the jumping process in equation form. We then assumed constancy of the coefficient of drainage density (k) and basin shape parameter (h) in basins of adjacent order, assumptions that were later confirmed in Chapter III to a high degree of accuracy. These assumptions enabled us to define the interdependence among the dimensionless ratios α, β, λ, and h and to write a polynomial defining the admissable region for the α,β-pairs for each value of h. The assumption concerning bifurcation further enabled us to define the most probable α,β-pairs within each admissable region. In Chapter III we related the concepts of this paper to the works of a number of other authors and examined in detail the validity of many of the relationships employed. Finally, in Chapter IV we gave our model a statistical structure by assuming that streams are randomly located and randomly oriented within the parent basin and that stream encounters along either axis follow

a Poisson probability law. The latter assumption simply implies that the number of encounters in a traverse is directly proportional to the distance travelled and that the distance between encounters follows an exponential probability law. Using the mean conditions described by these probability laws we were then able to write equations in terms of our dimensionless ratios which yield such vital statistics as numbers of streams of each order crossed, total length of streams of each order protruding into the crossing zone, distance between crossings, and to associate a confidence interval with each estimate.

As indicated earlier the model is designed to be part of a larger planning model. The model obviates the need for detailed map and aerial reconnaissance in a specific scenario area involving a large land mass and reduces a hitherto difficult problem to one that can be solved by a simple subroutine to the main planning program. It is emphasized, however, that without parameter quantification in the specific area of interest the model merely describes the most probable state. In this connection Chorley wrote "Often the achievement of exact equilibrium in nature occurs only momentarily as variations about the mean take place, and in these instances the existence of the steady state can only be recognized statistically" [Ref. 25:137]. As indicated by the sensitivity analysis, especially with respect to β, the variations about the equilibrium can be expected to be significant. Thus, it is important to

quantify as many of the parameters as possible in the area of interest. As a minimum, β should be so quantified, however, the model results can be made fully acceptable by quantifying both α and β. Overprinting topographic maps with this information along with the order of the parent stream in the scenario area could be accomplished without great difficulty. A traverse across a scenario area would likely encounter a number of different α and β values. This would merely require that the model equations be solved for each α and β value for the specific traverse distance applicable to each.

The model output would perhaps have its greatest utility when used in conjunction with a cost model and an effectiveness model in order to determine the best mix of stream crossing equipment for a unit of a given size operating in a preselected scenario area. The model would facilitate comparison of inventory mixes and mixes of hypothetical equipment projected for future development. The model could even be used to develop doctrine and to select routes of advance within fixed scenarios. The latter aspects should render the model quite useful in the conduct of large scale war games, especially those involving computerization. For example, what would be the trade-off value in attaching a bridge company to Force A versus Force B when the two forces were advancing along different routes? A question of this nature is likely to occur within any war game, and for many scenarios could best be answered

using the output of this model. The ability to answer questions of this nature without making weak assumptions and without the expense of actual stream counting should add a new dimension to war gaming, a dimension in which the entire war game would be more closely aligned with reality.

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13. ABSTRACT

A model has been developed for estimating the number, sizes, and interfluvial distances of streams to be crossed in traversing a naturally occurring drainage basin. A relationship among four network parameters has been found which, upon quantification, provide sufficient information to completely structure a mature drainage system. The four parameters are: stream length ratio, bifurcation ratio, basin shape parameter, and coefficient of drainage density. The model is amenable to computer solution. Procedures are described for parameter quantification using common topographic maps.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
BASIN MODEL						
DRAINAGE NETWORK MODEL						
DRAINAGE NETWORK PARAMETERS						
BIFURCATION RATIO						
STREAM LENGTH RATIO						
BASIN AREA RATIO						
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